Simultaneous Systems

Abi Adams

HT 2017

Recap

Aim: develop an understanding for the role of exclusion restrictions in identifying simultaneous systems, and the meaning of rank and order conditions for identification

1. Linear in means social interactions
2. The Cowles Model
3. Order & Rank Conditions

Recap

- Last lecture, reviewed the distinctions often made between causal and structural approaches to identification
- We found greater overlap between the two than some commentary might suggest
  - Causal approaches still embody structural assumptions to facilitate interpretation of parameters
  - Structural approaches often rely on independence assumptions and researchers should make an effort make their identification results more transparent

Today

- Applications involving simultaneity and interaction effects are one area in which structural insights are often required for identification and meaningful interpretation of parameters
- Causal approaches based on Rubin 'potential outcomes' framework typically postulate counterfactuals Y_0 and Y_1 without explicitly modelling the factors that determine these outcomes
- In addition to counterfactual invariance to the treatment assignment mechanism, the framework assumes:
  - No simultaneity in causal effects — outcomes cannot cause each other reciprocally
  - No social interactions or general equilibrium effects for objective outcomes

Applications

- This is clearly a drawback but one that can be dealt with using simple structural economic models rather than a quest alone for randomisation

Interesting Application: Peer Effects

Interesting Application: Peer Effects

- Endogeneous social effects: the propensity of an individual to behave in some way varies with the prevalence of some behaviour in a reference group containing the individual
- Crop up in many areas in economics....
  - Demand & Supply: how an individuals demand varies with price, when price is partly determined by aggregate demand in the relevant market of which the individual is a member of
  - Oligopoly: reaction functions wherein firm output is a function of industry output
  - Social norms and conformity effects in preferences

The Reflection Problem

- Peer effects lead to what Manski (1993) terms the **'Reflection Problem'**
- Problem similar to interpreting the simultaneous movements of a person and her reflection in the mirror — does the mirror image cause the person's movements or reflect them?
- Randomisation alone will not solve this problem

Model

- We start with a model based on Sacerdote (2001) in his study of peer effects in the context of Dartmouth College room mates
- Freshmen randomly assigned to dorms and to roommates so that we don't need to worry (yet!) about the problem of peers selecting each other based on observable and unobservable characteristics

Model

- Let student i's achievement, s_{ig}, depend on their ability, b_{ig}, and the performance and ability of their room mate

$$\begin{aligned} s_{1g} &= \theta_0 + \theta_1 b_{1g} + \theta_2 s_{2g} + \theta_3 b_{2g} + u_{1g} \\ s_{2g} &= \theta_0 + \theta_1 b_{2g} + \theta_2 s_{1g} + \theta_3 b_{1g} + u_{2g} \end{aligned} \tag{1}$$

  where each group $g = 1, ..., G$ is comprised of two individuals

- In Sacerdote's analysis, s_{ig} is first-year GPA and b is an index of ability observed for all applicants to Dartmouth

Model

$$s_{ig} = \theta_0 + \theta_1 b_{ig} + \theta_2 s_{jg} + \theta_3 b_{jg} + u_{ig} \tag{2}$$

- θ_2 captures endogenous peer effects
- θ_3 captures exogenous peer effects
- By random assignment, u_{ig} is independent of b_{jg}

Model

- Solving these simultaneously:

$$
\begin{aligned}
s_{1g} &= \theta_0 + \theta_1 b_{1g} + \theta_2 s_{2g} + \theta_3 b_{2g} + u_{1g} \\
&= \theta_0 + \theta_1 b_{1g} + \theta_3 b_{2g} + u_{1g} \\
&+ \theta_2 \left(\theta_0 + \theta_1 b_{2g} + \theta_2 s_{1g} + \theta_3 b_{1g} + u_{2g} \right) \\
(1 - \theta_2^2) s_{1g} &= \theta_0 (1 + \theta_2) + (\theta_2 \theta_3 + \theta_1) b_{1g} + (\theta_1 \theta_2 + \theta_3) b_{2g} \\
&+ (u_{1g} + \theta_2 u_{2g})
\end{aligned}
\tag{3}
$$

Model

- Solving these simultaneously:

$$s_{1g} = \frac{\theta_0(1+\theta_2)}{(1-\theta_2^2)} + \frac{(\theta_2\theta_3+\theta_1)}{(1-\theta_2^2)} b_{1g} + \frac{(\theta_1\theta_2+\theta_3)}{(1-\theta_2^2)} b_{2g} + \frac{(u_{1g}+\theta_2 u_{2g})}{(1-\theta_2^2)} \quad (4)$$

$$s_{1g} = \alpha + \beta b_{1g} + \gamma b_{2g} + v_{1g}$$

- Similarly,

$$s_{2g} = \alpha + \beta b_{2g} + \gamma b_{1g} + v_{2g} \quad (5)$$

- Given random assignment of room mates, and sufficient variation in ability, the reduced form coefficients are identified....

Model

- However, θ is not identified — we can estimate three parameters but we have four unknowns
- It is this point that we will get more precise about in this lecture
- Intuition: Abi & Brooke are room mates
  - Abi gets proper glasses that mean she can see properly, having a positive impact on her ability to study and her marks
  - Abi's increased test score has a positive effect on Brooke's due to peer group effects
  - However, again thanks to this effect, Brooke's improved performance has a positive impact on Abi, magnifying the original effect of just being able to see properly

Model

$$s_{1g} = \alpha + \beta b_{1g} + \gamma b_{2g} + v_{1g} \quad (6)$$

- While one cannot identify, θ, identification of certain structural features is possible
- Imagine that $\gamma = (\theta_1\theta_2 + \theta_3)/(1 - \theta_2^2) \neq 0$, then either θ_2 or θ_3 or both must be non-zero
- This confirms that peer effects are present even if it does not permit whether it is an endogenous or exogenous peer effect

Model

- Identification of the structural coefficients is made possible through **exclusion restrictions**
- Exclusion restrictions might flow from your theoretical model and then perhaps with exogenous variation provided experimentally
- For example, some room mates randomly chosen to receive rewards, x for their academic performance, which are independent of u_{1g} and u_{2g}

$$\begin{aligned} s_{1g} &= \theta_0 + \theta_1 b_{1g} + \theta_2 s_{2g} + \theta_3 b_{2g} + u_{1g} \\ s_{2g} &= \theta_0 + \theta_1 b_{2g} + \theta_2 s_{1g} + \theta_3 b_{1g} + \theta_4 x_{2g} + u_{2g} \end{aligned} \quad (7)$$

Model

$$
\begin{aligned}
s_{1g} &= \alpha + \beta b_{1g} + \gamma b_{2g} + \delta_1 x_{2g} + v_{1g} \\
s_{2g} &= \alpha + \beta b_{2g} + \gamma b_{1g} + \delta_2 x_{2g} + v_{1g}
\end{aligned} \tag{8}
$$

- From the reduced form coefficients,

$$
\begin{aligned}
\delta_1 &= \frac{\theta_4 \theta_2}{1 - \theta_2^2} \\
\delta_2 &= \frac{\theta_4}{1 - \theta_2^2} \\
\frac{\delta_1}{\delta_2} &= \theta_2
\end{aligned} \tag{9}
$$

General Model

- To generalise this discussion, let's now consider a general system and the conditions required for identification

$$\begin{aligned} Y\gamma_1 + Z\beta_1 + u_1 &= 0 \\ &\vdots \\ Y\gamma_N + Z\beta_N + u_N &= 0 \end{aligned} \tag{10}$$

  where $Y = (y_1, ..., y_N)$ is the $1 \times N$ vector of all endogenous variables and $Z = (z_1, ..., z_M)$ is the $1 \times M$ vector of explanatory variables

- This can be written compactly as

$$\Gamma Y + BX + U = 0 \tag{11}$$

  with E(U) = 0

General Model

- A distinction is made between complete and incomplete models
- A complete model produces a unique Y from a given (X, U)
- In this case, the model is complete when Γ is full rank

$$\begin{aligned} \Gamma^{-1}\Gamma Y + \Gamma^{-1}BX &= \Gamma^{-1}U \\ Y &= \Pi X + V \end{aligned} \tag{12}$$

- We will look at incomplete models next time in the context of binary games

General Model

- Let's consider identification of the first equation

$$Y\gamma_1 + Z\beta_1 + u_1 = 0$$
$$\gamma_{11}y_1 + \gamma_{12}y_2 + \cdots + \gamma_{1N}y_N + \beta_{11}z_1 + \cdots + \beta_{1M}z_M = 0 \quad (13)$$

- Natural normalisation given that y_1 is the dependent variable in this case

$$-y_1 + \gamma_{12}y_2 + \cdots + \gamma_{1N}y_N + \beta_{11}z_1 + \cdots + \beta_{1M}z_M = 0 \quad (14)$$

General Model

- We have $(N-1)+M$ parameters to identify
- When do we know if we have sufficient restrictions to do so?
- Let our prior knowledge about $\delta = (\gamma_1', \beta_1')$ be represented as the constraint matrix R_1

$$R_1\delta_1 = 0 \tag{15}$$

where R_1 is a $J \times (N+M)$ matrix of known constants (where we assume that rank(R_1) = J so that there are no redundant equations

Example

- Returning to the peer effects example

$$s_{1g} = \gamma_{12} s_{2g} + \beta_{11} b_{1g} + \beta_{12} b_{2g} + \beta_{13} x_{1g} + \beta_{14} x_{2g} + u_{1g} \tag{16}$$

- so

$$\delta_1 = (-1, \gamma_{12}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14})' \tag{17}$$

- Imagine x_{2g} didn't directly enter the attainment equation for room mate 1, as we speculated, then

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

Order Condition for Identification

- A **necessary** condition for the first equation to be identified is:

$$J_1 \geq N - 1 \tag{19}$$

- This means that we must have at least as many restrictions on the parameter vector as there are endogenous variables in the equation of interest to identify the structural parameters
- Simplicity makes it attractive as a tool for studying identification
- But beware, this is necessary but not sufficient

Rank Condition for Identification

- A **sufficient** condition for the first equation to be identified is:
$$\text{rank}\left(R_1 \begin{bmatrix} \Gamma \\ B \end{bmatrix}\right) = N - 1 \tag{20}$$
- The $J_1 \times N$ vector $[\Gamma B]' = [R_1\delta_1, R_1\delta_2, ..., R_1\delta_N]$
- Do the columns of this matrix form a linearly independent set?
- Intuitively, for each equation, the variables excluded from the equations must appear in at least one of the other equations

Rank Condition for Identification

- When the rank condition holds, it is useful to refine the sense in which the equation is identified
  - Just-identified: $J_1 = N - 1$ — if we were to drop a single restriction, identification would fail as the order condition would fail
  - Over-identified: $J_1 > N - 1$ — possible to drop one or more of the restrictions and still achieve identification

Concluding Thoughts

- Is a model really required? See Angrist, Graddy & Imbens (2000) for the interpretation of IV in a simultaneous system to see the difficulties involved
- Somewhat counter-intuitively, nonlinearities in models can sometimes help with identification. See Lewbel (2016)
- Finally, intuitively similar results hold for non-linear SEMs — see Rothenberg (1971)